The Z Transformation

By H. A. HELM

(Manuscript received April 15, 1958)

The Stieltjes integral is used to develop a rigorous derivation of the z transform. Sufficient properties of the transformation are included to form a reasonably complete basis for the operational solution of constant coefficient, linear, finite difference equations.

I. INTRODUCTION

The desire to use digital computers in automatic control loops created the need for methods with which to analyze systems that are partly continuous and partly discrete. Since the methods of network theory could be applied to the analysis of the continuous part of such a hybrid system, it was natural that such methods should be extended to include the discrete case. This resulted in the z transform introduced by Raggazini and Zadeh.[1] There is today an extensive literature devoted to the z transform.[2, 3, 4] However, the fundamental assumption of the z transform derivation is that the process of instantaneous sampling is equivalent to the amplitude modulation of a train of unit impulses by the "sampled" function. But the unit impulse as commonly defined has infinite height and zero width, and the process of amplitude modulating such a function is not intuitively clear. While it is true that such a process may be considered as an approximation to the behavior of a linear network with an amplifier and sampling switch, "impulse sampling" bears no simple relation to the manner in which the digital computer operates. The digital computer, in the type of real time operation typical of control system applications, works with sequences of numbers which represent a continuous function evaluated at particular instance of time. Since these numbers must of necessity be finite, "impulse sampling" is not an obvious mathematical model for describing the working of the computer. It is the intention of this paper to define the problem from the point of view of operations within the computer and to develop a rigorous and appealing derivation of the z transform.

In place of impulse modulation, the alternate approach is taken

of generalizing the definition of the Laplace transformation by means of the Stieltjes integral. This approach has the advantage of rigor and of more closely relating the operational solutions of continuous and discrete systems. As developed below, only rational transforms are considered. Also, in general, functions involving derivatives of impulses cannot be represented by the Stieltjes integral. Practically, these restrictions do not limit the applications greatly. Derivation of the principal properties of the z transform, based upon the Stieltjes integral definition, are given in the Appendix.

II. THE LAPLACE-STIELTJES TRANSFORMATION

The Stieltjes integral has the important property of including both sums and limits of sums (integrals). For the reader's convenience the definition of the Stieltjes integral as given in Widder[5] is repeated.

Let an interval (a, b) be divided into sub-intervals in the following manner:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

and let Δ equal the largest of these subintervals. Then the Stieltjes integral of $f(x)$ with respect to $\alpha(x)$ from a to b is

$$\int_a^b f(x)\, d\alpha(x) = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(\zeta_k)[\alpha(x_k) - \alpha(x_{k-1})], \tag{1}$$

where $x_{k-1} \leqq \zeta_k \leqq x_k$. The left-hand side of (1) is the usual notation for a Stieltjes integral. The integral itself is defined only when the limit on the right exists. It can be shown[5] that the integral exists if $f(x)$ is continuous and if $\alpha(x)$ is monotonic but not necessarily continuous, i.e., nonincreasing or nondecreasing. We shall assume that both these conditions apply to all functions to be considered. However, these conditions are quite strong and will be somewhat relaxed subsequently.

It is now possible to generalize the Laplace transformation by making the defining integral a Stieltjes integral. Thus,

$$L_s[f(t)] = \int_0^\infty f(t)e^{-st}\, d\alpha(t), \tag{2}$$

where

$$s = \sigma + j\omega.$$

It is assumed that (2) is subject to all the above restrictions. As defined by (2) the Laplace-Stieltjes transformation actually defines a different

transformation for each different selection of the function $\alpha(t)$. If $\alpha(t) = t$, (2) reduces to the usual definition of the Laplace transform. If $\alpha(t)$ is continuous and has a continuous derivative, (2) reduces to the Laplace transform of $\alpha'(t)f(t)$, which may be handled by the usual theorems of the Laplace transform. However, if $\alpha(t)$ is not continuous a new class of transformations results. For the purpose of this paper, the function $\alpha(t)$ will be defined as the "staircase" function which increases by unity at integral multiples of T but remains constant between such points. This function is shown in Fig. 1. The constant T is equivalent to the sampling period of modulation theory. From this point on, the function $\alpha(t)$ will be assumed to be that of Fig. 1. Thus, (2) may be evaluated for this $\alpha(t)$ by means of (1) as

$$\int_0^\infty f(t)e^{-st}\,d\alpha(t) = \sum_{n=0}^{\infty} f(nT)e^{-nTs} = F(e^{sT}), \tag{3}$$

where $\alpha(t)$ is given by Fig. 1, $s = \sigma + j\omega$ and $f(nT)$ is the function $f(t)$ evaluated at $t = nT$. Since $\alpha(t)$ is monotonic, and we have restricted $f(t)$ to continuous functions, (2) clearly *exists*. This does not imply that the series on the right converges or diverges. Since no part of $f(t)$ be-

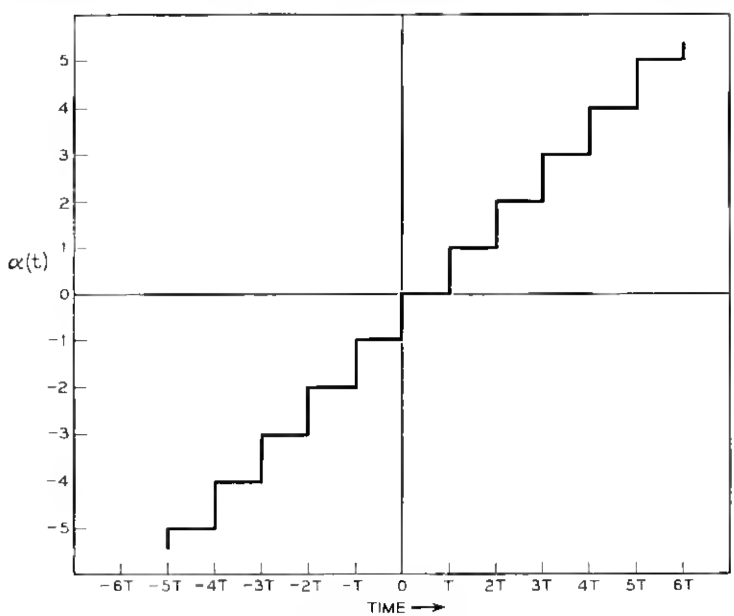

Fig. 1 — "Staircase" function $\alpha(t)$.

tween multiples of T affects (3), $f(t)$ need only be continuous and well defined in the neighborhood of the points $t = nT$; i.e., have no discontinuities or "jumps" at multiples of T.

A simple change of variable in (3) introduces the z transform. Let $z = e^{sT}$ and substitute in (3):

$$L_s[f(t)] = \int_0^\infty f(t)z^{-t/T}\,d\alpha(t) = \sum_{n=0}^{\infty} f(nT)z^{-n} = f(z). \tag{4}$$

The expression (4) emphasizes the power series nature of the z transform. Since the functions with which we shall deal are convergent, they can almost always be written in closed form. In fact, to allow the order of certain limit functions to be interchanged in the development of various theorems, absolute convergence of (4) will be assumed. That is,

$$\sum_{n=0}^{\infty} | f(nT)z^{-n} | \leqq M, \tag{5}$$

where M is finite although possibly very large. The magnitude of z is $e^{-nT\sigma}$, and if (5) holds for some σ, say σ_0, it will obviously hold for any $\sigma > \sigma_0$. In the following sections, the functions $f(t)$ are now restricted to those functions for which (5) holds. Rewriting (5) as a Stieltjes integral leads to:

$$\int_0^\infty | f(t) | e^{-\sigma t}\,d\alpha(t) \leqq M, \tag{5a}$$

which holds for the Laplace transform when $\alpha(t) = t$.[6] The restriction of absolute convergence has little practical effect upon the utility of the transform.

III. THE INVERSE TRANSFORM

An inverse transform is necessary for operational completeness. The derivation is straightforward, making use of Cauchy's method for evaluating the coefficients of a power series.[7] The proof proceeds from the definition of the z transform in (4):

$$L_s[f(t)] = \sum_{n=0}^{\infty} f(nT)z^{-n} = F(z).$$

Expanding,

$$\begin{aligned} f(z) = f(0) + f(T)z^{-1} + \cdots + f(nT - T)z^{-(n-1)} + f(nT)z^{-n} \\ + f(nT + T)z^{-(n+1)} + \cdots, \end{aligned} \tag{6}$$

which is absolutely convergent for $|z| \geqq e^{\sigma_0 T}$. Since $F(z)$ is a power series, absolute convergence also implies uniform convergence within the radius of convergence. Hence, (6) may be integrated term by term along a contour such that $|z| > e^{\sigma_0 T}$. Multiplying by z^{n-1} and so integrating gives:

$$\begin{aligned}\int_\Gamma z^{n-1}F(z)\,dz = \int_\Gamma f(0)z^{n-1}\,dz + \int_\Gamma f(T)z^{n-2}\,dz + \cdots \\ + \int_\Gamma f(nT - T)\,dz + \int_\Gamma f(nT)z^{-1}\,dz + \cdots \quad (7) \\ + \int_\Gamma f(nT + mT)z^{-(m+1)} \cdots,\end{aligned}$$

where $m = 1, 2, 3, \cdots$ and, Γ the contour of integration, is a circle enclosing the origin of the z plane and whose radius is greater than $e^{\sigma_0 T}$. It is obvious that all integrals except $\int_\Gamma f(nT)z^{-1}\,dz$ either have no singularity within the contour of integration, and hence are zero, or else have a pole at the origin of order greater than unity, and hence also are zero. Therefore, (7) reduces to

$$\int_\Gamma z^{n-1}F(z)\,dz = f(nT)\int_\Gamma z^{-1}\,dz, \tag{8}$$

since $f(nT)$ is the function $f(t)$ evaluated at $t = nT$, and hence a constant. Thus, the inversion formula becomes

$$f(nT) = \frac{1}{2\pi j}\int_\Gamma z^{n-1}F(z)\,dz. \tag{9}$$

It is necessary to add a word of caution here. Equation (9) only gives the value of the function at *one* point, $t = nT$. Obviously, by assigning any given integer to n the value at any point may be obtained and, usually, this is sufficient. However, in the case of certain summations this will lead to a very confusing notation. To avoid this, we introduce the notation

$$\{f(nT)\}_m$$

to indicate the sequence

$$f(0), f(T), f(2T) \cdots, f(nT), \cdots, f(mT).$$

Omission of the subscript m will denote an infinite sequence.

It is now possible to summarize the Laplace-Stieltjes pairs:

$$L_s[f(t)] = \int_0^\infty f(t)e^{-st}\,d\alpha(t) = F(e^{sT}) \tag{3}$$

or

$$L_s[f(t)] = \int_0^\infty f(t)z^{-t/T}\,d\alpha(t) = F(z), \tag{3a}$$

$$L_s^{-1}[F(z)] = \frac{1}{2\pi j}\int_\Gamma z^{n-1}F(z)\,dz = f(nT). \tag{9}$$

A closer inspection of the above pairs indicates a very interesting property of the Laplace-Stieltjes transformation. A continuous function $f(t)$ is transformed and then the inverse operation performed. However, the function is then only defined at integral multiples of T, i.e., at $t = nT$. This is exactly equivalent to "sampling"; that is, the computer, by means of some encoding device, evaluates instantly a continuous function of time (commonly represented by voltages, shaft positions, etc.) at periodic intervals. The L_s transformation is a mathematical model or abstraction which represents this process. It is very often the case that the function which is to be sampled (a voltage, for instance) is applied to a linear network and then sampled, and it would be very convenient to be able to analyze the system directly from the Laplace transform. Such results follow from the relationships between the Laplace and Laplace-Stieltjes transforms which we develop below.

IV. THE RELATIONSHIP BETWEEN THE LAPLACE AND LAPLACE-STIELTJES TRANSFORMATIONS

The transform pairs (3) or (3a) and (9) are sufficient to derive operational methods for discrete linear systems which are similar to those of continuous linear systems. However, there are several reasons for examining the relationship between the two transforms. In the first place, many of the most interesting problems arise from the analysis of systems which are partly continuous and partly discrete. Also, the relationship between the Laplace-Stieltjes and Laplace transforms may be expressed as a convolution integral, which historically was used first in the study of these systems and which is still a very handy computational formula.

Returning to (4), we have

$$L_s[f(t)] = \sum_{n=0}^{\infty} f(nT)e^{-nTs}. \tag{4}$$

The inverse of the normal Laplace transform, with the simple change of variable $t = nT$ and using p in place of the usual s, is

$$f(nT) = \frac{1}{2\pi j}\int_\Gamma F(p)e^{pnT}\,dp, \tag{10}$$

where Γ, the contour of integration, encloses the poles of $F(p)$, which is assumed rational. Substitution of (10) into (4) gives

$$\begin{aligned} L_s[f(t)] &= \sum_{n=0}^{\infty} e^{-nTs}\frac{1}{2\pi j}\int_\Gamma F(p)e^{pnT}\,dp \\ &= \frac{1}{2\pi j}\sum_{n=0}^{\infty}\int_\Gamma F(p)e^{(p-s)nT}\,dp. \end{aligned} \tag{11}$$

Since the series (4) and the integral are both uniformly convergent, the order of summation and integration may be interchanged:

$$L_s[f(t)] = \frac{1}{2\pi j}\int_\Gamma F(p)\sum_{n=0}^{\infty} e^{(p-s)nT}\,dp. \tag{12}$$

If $|e^{(p-s)T}| < 1$, we may sum the geometric series in (12) to

$$\sum_{n=0}^{\infty} e^{(p-s)nT} = \frac{1}{1-e^{(p-s)T}}, \tag{13}$$

where the condition that this summation be valid is that $\operatorname{Re} p < \operatorname{Re} s$. Thus, (12) becomes

$$L_s[f(t)] = \frac{1}{2\pi j}\int_\Gamma \frac{F(p)}{1-e^{(p-s)T}}\,dp. \tag{14}$$

The contour of integration, Γ, is the usual one parallel to the imaginary axis extending from $-j\infty$ to $+j\infty$ to include all possible poles. However, since we do not wish to exclude functions which do not vanish as s(or p) approaches infinity [in particular, $F(p) = \text{constant}$] the contour is closed by an infinite semicircle to the left from $+j\infty$ to $-j\infty$. The requirement that $\operatorname{Re} p < \operatorname{Re} s$ is equivalent to stating that Γ shall include the poles fo $F(p)$ but exclude the poles of

$$\frac{1}{1-e^{(p-s)T}}.$$

If the inverse Laplace transform of

$$F(s) = \frac{1}{1-e^{-sT}}$$

be interpreted as the sum of a sequence of unit impulses, $\delta_T(t)$, a distance T apart, then the amplitude modulation of $\delta_T(t)$ by some function $f(t)$ may be found, from the complex convolution formula of Laplace transform theory, to be:

$$L[f(t)\delta_T(t)] = \frac{1}{2\pi j} \int_\Gamma \frac{F(p)}{1 - e^{(p-s)T}} \, dp,$$

which is, of course, the Laplace-Stieltjes transform of $f(t)$ as given by (14). Hence, the Laplace-Stieltjes transform is formally equivalent to the results of impulse modulation in the sense that both lead to the same transform. However, the definition of the Laplace-Stieltjes transform is rigorous and it directly relates discrete and continuous systems. Intuitively, one would expect that, as the interval between samples approaches zero, the Laplace-Stieltjes transform should approach the Laplace; i.e., the discrete system should look more like the continuous. This follows from the definition of the Laplace-Stieltjes transform. Note that, for the $\alpha(t)$ of Fig. 1,

$$\lim_{T \to 0} T\alpha(t) = t.$$

Thus,

$$\lim_{T \to 0} TL_s[f(t)] = \lim_{T \to 0} \int_0^\infty f(t)e^{-st} \, dT\alpha(t)$$

$$= \int_0^\infty f(t)e^{-st} \, dt = L[f(t)],$$

which is the desired relation, the Laplace-Stieltjes transformation approaching the Laplace in the same manner as the staircase distribution function $T\alpha(t)$ approaches the straight line t.

Equation (14) is a very useful computational tool and can be used to prepare a table of Laplace-Stieltjes transforms from the common tables of Laplace transforms. Some elementary functions are given in Table I, where a direct comparison between the two transforms can be made, the Laplace-Stieltjes transform being written in the e^{sT} form. Such a comparison is interesting, but the relationship between the two transforms can be better shown by a closer examination of the transforms of some elementary functions. Consider first $f(t) = e^{-\alpha t}$:

$$L(e^{-\alpha t}) = \frac{1}{s + \alpha} = F(s), \tag{15}$$

$$L_s(e^{-\alpha t}) = \frac{e^{sT}}{e^{sT} - e^{-\alpha T}} = F(e^{sT}). \tag{16}$$

The single pole at $s = -\alpha$ of $F(s)$ is shown in Fig. 2(a) in the usual manner. However, $F(e^{sT})$ has an infinite number of singularities occurring at $s = -\alpha \pm 2\pi n/T (n = 0, 1, 2, \cdots)$. Thus, the effect of sampling is to

multiply the single real pole of the L transform into an infinite number of complex poles as shown in Fig. 2(b). The case of imaginary roots in the s plane is even more interesting. Let $f(t) = \sin \beta t$.

$$L[\sin \beta t] = \frac{\beta}{s^2 + \beta^2} = F(s), \tag{17}$$

$$L_s[\sin \beta t] = \frac{e^{sT} \sin \beta T}{e^{2sT} - 2e^{sT} \cos \beta T + 1}. \tag{18}$$

The singularities of $F(s)$ are shown in Fig. 3(a) and those of $F(e^{sT})$ in Fig. 3(b). Here, the sampling multiplies the original *pair* of poles into an infinite number of such pairs. The center of each pair is separated from the next by a distance of $2\pi/T$. The distance or period T may be identified with a radian sampling frequency $\omega_s = 2\pi/T$. From Fig. 3(b) it is apparent that, if $\beta \geqq \omega_s/2$, the pairs of poles overlap each other and form a new configuration which is indistinguishable from a configuration resulting from some function of radian frequency less than $\omega_s/2$. This result also follows from Shannon's sampling theorem which, in effect, states that, if the original signal is to be recovered after sampling, then the sampling frequency must be greater than twice the highest frequency sampled.

If the L transformation of a function has a singularity in the right half of the s plane, the L_s transformation will have an infinite number in the right half plane.

TABLE I

$f(t)$	$F(s)$	$f(nT)$	$F(e^{sT})$
$u(t)$	$\frac{1}{s}$	$u(nT)$	$\frac{e^{sT}}{e^{sT} - 1}$
$e^{-\alpha t}$	$\frac{1}{s + \alpha}$	$e^{-\alpha nT}$	$\frac{e^{sT}}{e^{sT} - e^{-\alpha T}}$
$\sin \beta t$	$\frac{\beta}{s^2 + \beta^2}$	$\sin \beta nT$	$\frac{e^{sT} \sin \beta T}{e^{2sT} - 2e^{sT} \cos \beta T + 1}$
$e^{-\alpha t} \sin \beta t$	$\frac{\beta}{(s + \alpha)^2 + \beta^2}$	$e^{-\alpha nT} \sin \beta nT$	$\frac{e^{(s+\alpha)T} \sin \beta T}{e^{2(s+\alpha)T} - 2e^{(s+\alpha)T} \cos \beta T + 1}$
$e^{-\alpha t} \cos \beta t$	$\frac{s}{(s + \alpha)^2 + \beta^2}$	$e^{-\alpha nT} \cos \beta nT$	$\frac{(e^{(s+\alpha)T} - \cos \beta T)e^{(s+\alpha)T}}{e^{2(s+\alpha)T} - 2e^{(s+\alpha)T} \cos \beta T + 1}$

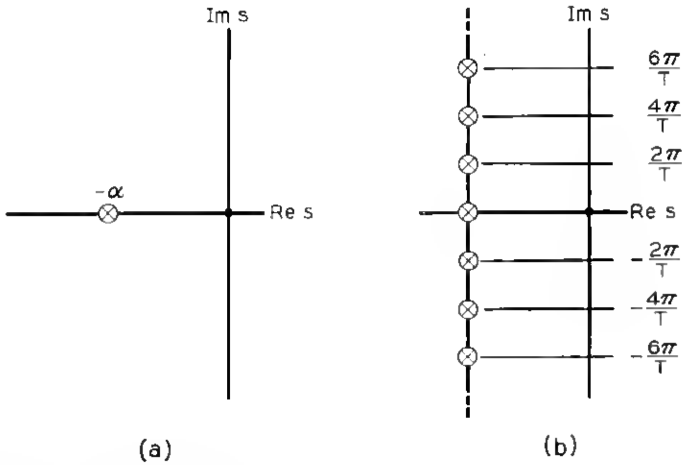

Fig. 2 — (a) Single S plane pole of $F(s) = 1/(s + \alpha)$; (b) infinite number of complex S plane poles of $F(e^{sT}) = e^{sT}/(e^{sT} - e^{-\alpha T})$.

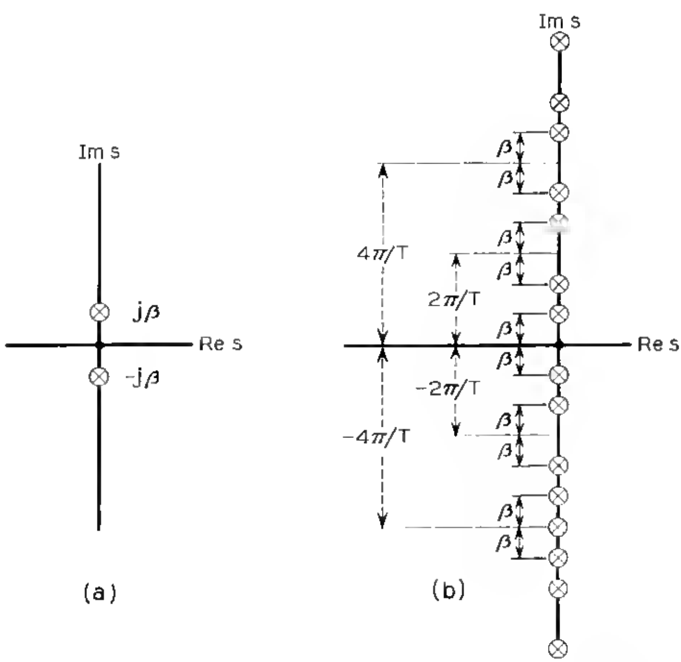

Fig. 3 — Pair of S plane poles of $F(s) = \beta/(s^2 + \beta^2)$; (b) infinite number of S plane pole pairs of $F(e^{sT}) = e^{sT} \sin \beta T/(e^{2sT} - 2e^{sT} \cos \beta T + 1)$.

The change of variable, $z = e^{sT}$, which introduces the z transform, simplifies the above mappings. It is well known that the transformation, $z = e^{sT}$, maps the imaginary axis of the complex s plane into the unit circle about the origin in the z plane. The left half of the s plane maps wholely within the unit circle, the right half maps exterior to it. Since the z plane mapping is repetitive for multiples of $2\pi/T$, the infinite number of roots and root-pairs in the s plane map into single roots and root-pairs in the z plane.

V. APPLICATION OF THE L_s TRANSFORMATION TO THE SOLUTION OF FINITE DIFFERENCE EQUATIONS

Many of the concepts of sampling can be applied to the solution of linear *finite difference* equations, with constant coefficients. These equations are simply linear combinations of sequences of numbers shifted forward and backward in time by integral multiples of some fixed interval. In the case of a digital computer operating in a control loop, the sequences are actually generated by sampling some continuous function of time. If the assumption is made that all the sequences in a finite difference equation result from such sampling, then the L_s transformation offers a very useful method for the operational solution of such an equation. The resulting solution is the "smooth" curve which, when sampled, will give the sequence satisfying the difference equation.

A finite difference may be defined in either of two ways. One could be called a *backward* difference, defined as

$$\Delta_b\{y(nT)\} = \{y(nT)\} - \{y(nT - T)\}. \tag{19}$$

That is, the backward difference is simply the difference of two sequences, one of which is the other shifted backward one interval in time. Since there is no possibility of ambiguity, the braces may be omitted and (19) written in the more usual form

$$\Delta_b y_{nT} = y_{nT} - y_{nT-T}\,. \tag{19}$$

In similar fashion, the *forward* difference may be written as:

$$\Delta_f y_{nT} = y_{nT+T} - y_{nT}\,. \tag{20}$$

Higher differences of course are formed by taking "differences of differences." That is,

$$\Delta_b{}^n y_{nT} = \Delta_b{}^{n-1} y_{nT} - \Delta_b{}^{n-1} y_{nT-T}$$

or

$$\Delta_f{}^n y_{nT} = \Delta_f{}^{n-1} y_{nT+T} - \Delta_f{}^{n-1} y_{nT}\,.$$

In order to eliminate possible confusion between differences, a difference equation can always be expanded and written in the ordinate form:

$$\sum_{j=1}^{n} b_j y_{nT+jT} + \sum_{i=0}^{m} a_i y_{nT-iT} = x_{nT} .$$

If the assumption is made that the sequences $\{y(nT)\}$, $\{x(nT)\}$ result from a sampling of some continuous function which has an L_s transform, then the finite difference equations of the above form are readily solved by the L_s transformation. This follows from the results of Property I of the L_s transform, which is proved in the Appendix. The property is repeated below without proof.

Property I: If $L_s[f(t)] = F(z)$ and a is a nonnegative integer, then:

$$\begin{aligned} L_s[f(t - aT)] &= L_s[\{f(nT - aT)\}] \\ &= z^{-a}\left[F(z) + \sum_{m=1}^{a} f(-mT)z^{m}\right] \end{aligned} \tag{21}$$

and

$$\begin{aligned} L_s[f(t + aT)] &= L_s[\{f(nT + aT)\}] \\ &= z^{a}\left[F(z) - \sum_{m=0}^{a-1} f(mT)z^{-m}\right]. \end{aligned} \tag{22}$$

Application of (21) to the *backward* difference equation (14) leads to

$$L_s[\Delta_b Y_{nT}] = \frac{z - 1}{z} Y(z) - y(-T) \tag{23}$$

and to the forward difference

$$L_s[\Delta_f t_{nT}] = (z - 1)Y(z) - zy(0). \tag{24}$$

The terms $y(0)$ in (23) and $y(-T)$ in (24) are the usual initial conditions, and allow the specification of arbitrary boundary conditions in a manner completely analogous to the insertion of initial conditions in the solution of differential equations. However, for simplicity, zero initial conditions will be assumed in the problem below.

As an example, the above can be applied to the simultaneous difference equations:

$$\begin{aligned} 0.25y_{n-1} + w_n &= x_n , \\ -1.0y_n - 0.25y_{n-1} + w_n + 0.5w_{n-1} &= 0, \end{aligned} \tag{25}$$

with T taken as unity for convenience. Letting

$$Y(z) = L_s[y_n],$$
$$W(z) = L_s[w_n],$$
$$X(z) = L_s[x_n],$$

and performing the indicated operation on (25) leads to:

$$\begin{aligned} 0.25Y(z) + zW(z) &= zX(z), \\ -(z + 0.25)Y(z) + (z + 0.5)W(z) &= 0, \end{aligned} \tag{26}$$

whence

$$W(z) = \frac{z(z + 0.25)X(z)}{z^2 + 0.5z + 0.125}. \tag{27}$$

As a "test function" let $x(n)$ be the *unit sample* defined as unity for $n = 0$ and zero elsewhere. The L_s transform of x_n is then $L_s[x_n] = 1$, and it follows that

$$W(z) = \frac{z(z + 0.25)}{[(z + 0.25)^2 + .0625]}. \tag{28}$$

By the application of (9),

$$w_n = 4(0.35)^{n+1} \sin (n + 1) \frac{3\pi}{4} + (0.35)^n \sin n \frac{3\pi}{4}. \tag{29}$$

In a similar manner w_n(or y_n) may be determined for any function x_n which is L_s-transformable.

In electrical network theory based upon the Laplace transform, the weighting function of a network is the time response of that network to a unit *impulse*. The analogy to the above response of a difference equation to a unit sample is clear. The fact that the difference equations could describe the operation of a digital computer performing linear operations in real time lends physical reality to the analogy and introduces the concept of a digital network. The designer of a servomechanism or feedback amplifier is concerned that his device shall be stable. The designer of a program for a digital computer is likewise concerned that his machine behave in a stable manner. Here, stability is defined in the sense of the electronic network designer that, for any bounded input, the output shall not continually increase. This is more elegantly and precisely stated in terms of the complex s plane. Here, the criterion for stability is that the characteristic equation have no roots in the right half of the s plane. Since the defining transformation $z = e^{sT}$ of the z plane maps the right

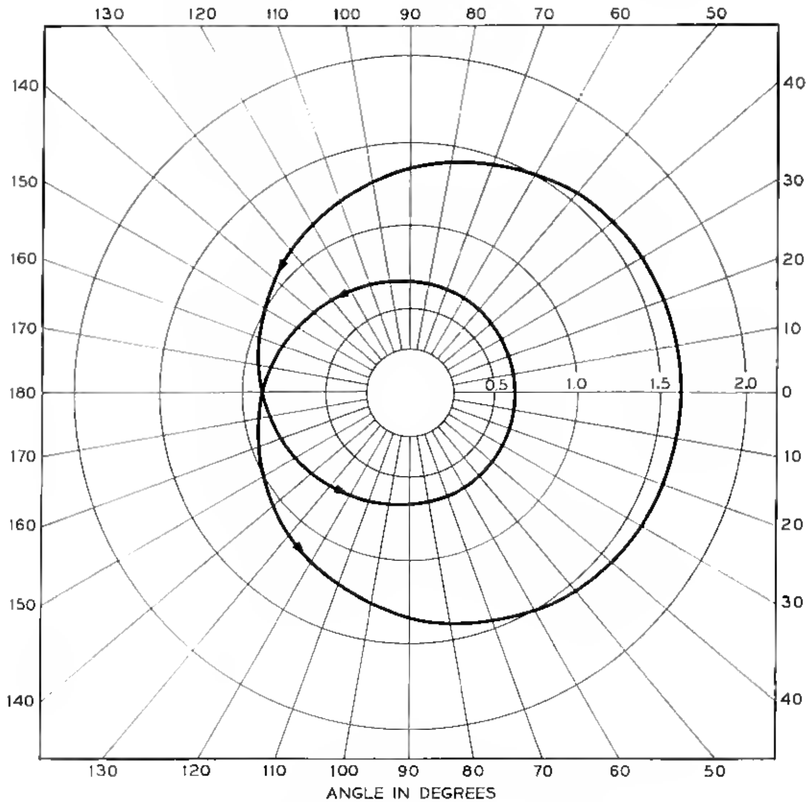

Fig. 4 — Nyquist plot of $z^2 + 0.5z + 0.125$.

half of the s plane onto the exterior of the unit circle, the criterion for stability of finite difference equations is that the characteristic equation shall have no roots exterior to the unit circle in the z plane. Intuitively, it would seem that Nyquist's criterion could be applied in one form or another directly in the z plane to determine stability. However, the application is not as attractive as in the continuous case. The finite difference equations (25) will be used to illustrate the point. As above, the roots of the denominator of (27), the L_s transformation of (25), determine the stability of the difference equations. Since all roots do indeed lie within the unit circle, as z takes on values along the unit circle in the *positive sense*, the plot of $z^2 + 0.5z + 0.125$ will encircle the origin in the *positive sense* a number of times equal to the number of zeros of the polynomial which lie within the unit circle (two in this case). Fig. 4 is such a plot and it is readily apparent that there are two encirclements.

In principle, the method can be extended to a polynomial of any order. As a practical matter, counting the number of encirclements from a polynomial of high order without making a mistake would be difficult. However, this is not the most important point. One of the great attractions of the Nyquist criterion to the practical designer of feedback devices is that not only does it determine stability but it also indicates at a glance the margins against instability. To know the allowable variation in the gain of an amplifier, as a specific example, is of great value to the designer (and manufacturer) of a feedback amplifier. Unfortunately, in the discrete case no such information is apparent and the indication is simply one of "go" or "no go".

If the change of variable $w = z^{-1} = e^{-sT}$ is made, the *right* half of the s plane is now mapped into the interior of the unit circle, and hence the criterion for stability becomes the more usual one of not enclosing the origin. Illustrating, the characteristic equation of (27) becomes;

$$w^{-2} + 0.5w^{-1} + 0.125. \tag{30}$$

As w takes on values on the unit circle in the positive sense, (30) has exactly the same values as shown in Fig. 4 with the exception that the curve now encloses the origin in the *negative* sense and, hence, stability depends also upon the *sense* of enclosure. The difficulties above will usually require that stability analysis still be made in the s plane. However, some advantage can be taken of the angle preserving properties of the change of variable $s = 1/T \ln z$.

VI. CONCLUSION

The Laplace-Stieltjes derivation of the z transform is straightforward and rigorous. As a mathematical model of the sampling process it has the advantage of also describing some of the operations possible within the computer. In particular, since it can be used as a basis for the operational solution of linear finite difference equations it closely relates the solution of discrete linear systems and continuous linear systems. It is of considerable advantage to be able to apply the methods of network analysis to this type of computer operation. The Laplace-Stieltjes transformation forms the connection in a very clear manner.

VII. ACKNOWLEDGMENT

The author would like to express his sincere appreciation to J. G. Tryon of Bell Telephone Laboratories for his constant encouragement

and help and to N. J. Rose of the Stevens Institute of Technology for the proof of the property of the transform for a finite sum.

APPENDIX

PROPERTIES OF THE L_s TRANSFORMATION

The utility of the L_s transform is increased by various properties pertaining to its use. The more important of these properties are derived below with some discussion of the area of application. Such discussion must of necessity be brief.

Real Translation

This property is the basis upon which the operational solution of linear finite difference equations is based.

Property I: If $L_s[f(t)] = F(z)$ *and a is a nonnegative integer, then:*

$$L_s[f(t - aT)] = z^{-a}\left[F(z) + \sum_{m=1}^{a} f(-mT)z^{m}\right],$$

and

$$L_s[f(t + aT)] = z^{a}\left[F(z) - \sum_{m=0}^{a-1} f(mT)z^{-m}\right].$$

The proof of the first part follows:

By definition

$$L_s[f(\tau)] = \int_0^\infty f(\tau)e^{-s\tau}\, d\alpha(\tau) = F(e^{sT}).$$

Dividing the range of integration into two parts:

$$F(e^{sT}) = \int_{-aT}^{\infty} f(\tau)e^{-s\tau}\, d\alpha(\tau) - \int_{-aT}^{0-} f(\tau)e^{-s\tau}\, d\alpha(\tau).$$

We now let $\tau = t - aT$ in the first integral on the right-hand side:

$$F(e^{sT}) = \int_0^\infty f(t - aT)e^{-s(t-aT)}\, d\alpha(t - aT) - \int_{-aT}^{0-} f(\tau)e^{-s\tau}\, d\alpha(\tau). \quad (31)$$

From Fig. 1 it is apparent that

$$d\alpha(t - aT) = d\alpha(t).$$

Hence,

$$F(e^{sT}) = e^{aTs}\int_0^\infty f(t - aT)e^{-st}\, d\alpha(t) - \int_{-aT}^{0-} f(\tau)e^{-s\tau}\, d\alpha(\tau). \quad (32)$$

But the second integral may be evaluated as:

$$\int_{-aT}^{0-} f(\tau)e^{-s\tau}\,d\alpha(\tau) = \sum_{m=1}^{a} f(-mT)e^{mTs}. \tag{33}$$

Substitution of (33) into (32) and rearrangement leads to

$$e^{-aTs}\left[F(e^{sT}) + \sum_{m=1}^{a} f(-mT)e^{mTs}\right] = \int_0^{\infty} f(t - aT)e^{-st}\,d\alpha(t).$$

But the right-hand side of the above is by definition $L_s[f(t - aT)]$. Making the usual substitution of $z = e^{sT}$ now leads to the desired result:

$$L_s[f(t - aT)] = z^{-a}\left[F(z) + \sum_{m=1}^{a} f(-mT)z^{m}\right]. \tag{34}$$

We note in particular that, for $a = 1$,

$$L_s[f(t - aT)] = z^{-1}[F(z) + f(-T)].$$

For proof of the second part, the range of integration is divided into two parts:

$$L_s[f(\tau)] = \int_{aT-}^{\infty} f(\tau)e^{-s\tau}\,d\alpha(\tau) + \int_0^{(a-1)T+} f(\tau)e^{-s\tau}\,d\alpha(\tau).$$

Letting $\tau = t + aT$ in the first integral on the right-hand side, we have

$$F(e^{sT}) = \int_{0-}^{\infty} f(t + aT)e^{-s(t+aT)}\,d\alpha(t + aT) + \int_0^{(a-1)T+} f(\tau)e^{-s\tau}\,d\alpha(\tau).$$

Rearranging and substituting $d\alpha(t + aT) = d\alpha(t)$,

$$e^{aTs}\left[F(e^{sT}) - \int_0^{(a-1)T+} f(\tau)e^{-st}\,d\alpha(t)\right] = \int_{0-}^{\infty} f(t + aT)e^{-s\tau}\,d\alpha(t). \tag{35}$$

The integral on the left is obviously

$$\int_0^{(a-1)T+} f(\tau)e^{-s\tau}\,d\alpha(\tau) = \sum_{m=0}^{a-1} f(mT)e^{-mTs},$$

and substitution into (35) gives

$$e^{aTs}\left[F(e^{sT}) - \sum_{m=0}^{a-1} f(mT)e^{-mTs}\right] = L_s[f(t + aT)],$$

whence

$$L_s[f(t + aT)] = z^{a}\left[F(z) - \sum_{m=0}^{a-1} f(mT)z^{-m}\right]. \tag{36}$$

For the special case of $a = 1$, we have

$$L_s[f(t + T)] = z[F(z) - f(0)] = L_s[f(t + T)]. \tag{37}$$

Finite Differences

An immediate consequence of the real translation property is that for finite differences. If finite differences are defined as in the text, we have:

Property II: If the sequence $\{f(nT)\}$ resulting from the sampling of the continuous function $f(t)$ has the L_s transform $F(z)$, then:

$$L_s[\Delta_b\{f(nT)\}] = L_s[\Delta_b f_{nT}] = \frac{z-1}{z} F(z) - f(-T),$$

$$L_s(\Delta_f f_{nT}) = (z - 1)F(z) - zf(0).$$

By definition,

$$L_s(\Delta_b f_{nT}) = L_s(f_{nT} - f_{nT-T}).$$

By linearity of the transform and (34),

$$\begin{aligned} L_s(\Delta_b f_{nT}) &= F(z) - z^{-1}[F(z) + f(0)] \\ &= \left(\frac{z-1}{z}\right) F(z) - f(-T). \end{aligned} \tag{38}$$

Again, by definition,

$$L_s(\Delta_f f_{nT}) = L_s(f_{nT+T} - f_{nT})$$

and, by linearity of the transform and (37),

$$L_s(\Delta_f f_{nT}) = z[F(z) - f(0)] - F(z) = (z - 1)F(z) - zf(0). \tag{39}$$

Finite Summation

As integration is the inverse operation of differentiation, summation can be considered the inverse operation of taking differences. This is demonstrated more clearly in the property below. The process of finite summation is best demonstrated in the case of a computer operating in real time. The computer samples a function which is continuous and well defined at the sampling instants. At each sample the computer adds that sample to the sum of all preceding samples. If the result of this operation is a sequence $\{g(nt)\}$, we have as the value of $g(nt)$ at any time nT:

$$g(nT) = \sum_{k=0}^{n} f(kT),$$

where $\{f(kT)\}_k$ is the sequence of sampled inputs. The analogy to integration with respect to time is clear. The L_s transform of such a summation is given below.

Property III: If $\{g(nT)\}$ is an infinite sequence such that each value of it is given by $g(nT) = \sum_{k=0}^{n} f(kT)$, then $G(z) = z/(z - 1)F(z)$, where $F(z)$ is the L_s transform of the sequence $\{f(nT)\}$.

By definition:

$$G(z) = \sum_{n=0}^{\infty} g(nT)z^{-n}.$$

Substitution of the value for $g(nT)$ gives

$$G(z) = \sum_{n=0}^{\infty} z^{-n}\left[\sum_{k=0}^{n} f(kT)\right].$$

Since we are dealing with uniformly convergent series, the order of summations may be interchanged, provided a suitable change in the limits is made in a manner equivalent to the change in limits when the order of integration is interchanged in double integration. Thus,

$$G(z) = \sum_{k=0}^{\infty} f(kT) \sum_{n=k}^{\infty} z^{-n},$$

which may be written as

$$G(z) = \sum_{k=0}^{\infty} f(kT)z^{-k} \sum_{n=0}^{\infty} z^{-n}.$$

However, the series in n may be summed as $1/(1 - z^{-1})$, and hence

$$G(z) = \sum_{k=0}^{\infty} f(kT)z^{-k}\left[\frac{z}{z-1}\right],$$

and hence

$$G(z) = \frac{z}{z-1} F(z). \tag{40}$$

Complex Multiplication

The superposition property of electrical networks is a very elegant and useful result of their linearity. For continuous linear networks, superposition is most concisely represented as a convolution integral, which has a particularly important Laplace transform. The same ideas also apply in the discrete case with the integral replaced by a summation.

Property IV: If $f(t)$ and $w(t)$ have the L_s transforms $F(z)$ and $W(z)$, then:

$$W(z)F(z) = L_s\left[\sum_{k=0}^{n} w(kT)f(nT - kT)\right].$$

By definition,

$$W(z) = \sum_{k=0}^{\infty} w(kT)z^{-k},$$

$$W(z)F(z) = \sum_{k=0}^{\infty} w(kT)z^{-k}F(z),$$

but

$$z^{-k}F(z) = L_s[f(t - kT)] = L_s\{[f(nT - kT)]\}.$$

Hence,

$$\begin{aligned} W(z)F(z) &= \sum_{k=0}^{\infty} w(kT)L_s[f(t - kT)] = \sum_{k=0}^{\infty} L_s[w(kT)f(t - kT)] \\ &= L_s\left[\sum_{k=0}^{\infty} w(kT)f(t - kT)\right]. \end{aligned} \tag{41}$$

At time $t = nT$ we have

$$W(z)F(z) = L_s\left[\sum_{k=0}^{\infty} w(kT)f(nT - kT)\right]$$

but $f(t) = 0$; $t < 0$ and therefore

$$W(z)F(z) = L_s\left[\sum_{k=0}^{n} w(kT)f(nT - kT)\right]. \tag{42}$$

Scale Change

Property V: If $L_s[f(t)] = F(z)$, then $L_s[e^{-at}f(t)] = F(kz)$, where $k = e^{+aT}$.
From definition,

$$\begin{aligned} L_s[e^{-aT}f(t)] &= \int_0^{\infty} f(t)e^{-(sTa)T}\, d\alpha(t) \\ &= \sum_{n=0}^{\infty} f(nT)[e^{-aT}e^{-sT}]^n, \end{aligned}$$

$$L_s[e^{-aT}f(t)] = \sum_{n=0}^{\infty} f(nT)[kz]^{-n} = F(kz). \tag{43}$$

REFERENCES

1. Ragazzini, J. R. and Zadeh, L. A., The Analysis of Sampled Data Systems, Trans. A.I.E.E., **71**, Part II, November 1952, pp. 225–234.
2. Barker, R. H., The Pulse Transfer Function and Its Application to Sampling Servo-Systems, Institution of Electrical Engineers (British) Monograph No. 43, July 1952.
3. Truxall, J. G., *Automatic Feedback Control System Synthesis*, McGraw-Hill, New York, 1955.
4. Jury, E. I., Analysis and Synthesis of Sampled Data Control Systems, Trans. A.I.E.E., **73**, Part I, September 1954, pp. 332–346.
5. Widder, D. V., *Advanced Calculus*, Prentice-Hall, New York, 1942.
6. Gardner, M. F. and Barnes, J. L., *Transients in Linear Systems*, John Wiley & Sons, New York, 1942.
7. Widder, D. V., *The Laplace Transform*, Princeton University Press, Princeton, N. J., 1946.